

THE COAREA FORMULA, CONDITION (N) AND RECTIFIABLE SETS FOR SOBOLEV FUNCTIONS ON METRIC SPACES

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ABSTRACT. The purpose of this paper is to study measure-theoretic properties of functions u belonging to a vector-valued Sobolev class on metric measure spaces that admit a Poincaré inequality and are equipped with a doubling measure. The properties we have selected to study are those that are closely related to those in establishing a coarea formula for u . In particular, we show that both u and the graph mapping of u satisfy Luzin's condition (N). Moreover, it is shown that the graph of u is countably Hausdorff rectifiable and that the mapping u satisfies the coarea formula.

1. INTRODUCTION

The main objective of this note is to study some measure-theoretic properties of mappings belonging to the Newtonian space, $N^{1,p}(X; \mathbb{R}^m)$, which is the analogue of the vector-valued Sobolev space $W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$. Here X is a metric measure space that possesses a measure, μ , that is doubling and that supports a Poincaré inequality. With this structure imposed on X , we establish Luzin's condition (N), the countable Hausdorff rectifiability of the graph of u , and the coarea formula, Theorem 6.1. We are thus able to virtually replicate the main results of [24], in which the following coarea formula was established for mappings, u , in $W^{1,p}(\Omega; \mathbb{R}^m)$, namely,

Theorem 1.1. *Suppose that $1 \leq m \leq n$, Ω is an open set in \mathbb{R}^n , and that $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ where either $p > m$ or $p \geq m = 1$. Then $u^{-1}(y)$ is countably \mathcal{H}^{n-m} rectifiable for almost all $y \in \mathbb{R}^m$ and*

$$(1.1) \quad \int_{\Omega} g(x) |J_m u(x)| dx = \int_{\mathbb{R}^m} \int_{u^{-1}(y)} g(x) d\mathcal{H}^{n-m}(x) dy,$$

where $g : \Omega \rightarrow \mathbb{R}$ is integrable and $J_m u$ is the m -dimensional Jacobian. Recall that $|J_m u|$ is the square root of the sum of the squares of the determinants of the m by m minors of the differential of u .

2000 *Mathematics Subject Classification.* Primary: 46E35, 49Q15; Secondary: 28A99, 43A85.

Key words and phrases. Coarea formula, coarea property, condition (N), doubling measure, graph, Luzin's condition, metric space, Newtonian space, Poincaré inequality, rectifiability, Sobolev space, upper gradient.

Niko Marola supported by the Academy of Finland and Emil Aaltosen säätiö.

Remark 1.2. In the above statement of the theorem, it is assumed that f is quasicontinuous, see Section 2 below. As demonstrated in [24], the requirement $p > m$ is necessary. Moreover, when $p > m$, if it is assumed that ∇u belongs to the Lorentz space, $L^{m,1}$, which is a slightly smaller space than L^m , then the coarea formula (1.1) still holds.

Recall that a set in a metric space is countably Hausdorff rectifiable if it can be covered, up to Hausdorff measure negligible sets, by a countable family of Lipschitz images of subsets of \mathbb{R}^m . For the study of rectifiable sets in metric and Banach spaces, see, e.g., Ambrosio–Kirchheim [1]. Also recall that a mapping $f : \Omega \rightarrow \mathbb{R}^m$, Ω is an open set in \mathbb{R}^m , is said to satisfy Luzin’s condition (N) if $\mathcal{L}^m(f(E)) = 0$ whenever $E \subset \Omega$ and $\mathcal{L}^m(E) = 0$, where \mathcal{L}^m denotes the Lebesgue measure. Condition (N) for Sobolev functions in the Euclidean case has been studied, for instance, in Martio–Ziener [25], Malý–Martio [23], Malý et al. in [24], and Reshetnyak [28]; see also the references in these papers. For a discussion in metric measure spaces consult Heinonen et al. [15].

The paper is organized as follows. The second section introduces the necessary background material such as the definition of doubling measures, upper gradients, Poincaré inequality, Newtonian spaces, i.e. Sobolev spaces on metric spaces, and capacity. In the third section we establish a general criterion for condition (N) in the spirit of Radó and Reichelderfer [27, V.3.6], see also Malý et al. [24]. We also record the estimates that pair off the capacity and the Hausdorff content. We close Section 3 by proving that the graph mapping of a vector-valued Newtonian function satisfies condition (N). In Section 4, in addition to studying rectifiability properties of the graphs of a Newtonian function, we prove that elements in the Newtonian class satisfy the coarea property, which is sometimes referred to as condition (co-N). In Section 5 we deal with the coarea formula for vector-valued Newtonian functions and we close this note by stating an observation on absolute continuity of Newtonian functions in the spirit of Malý [21] and show, moreover, that Newtonian functions satisfy condition (N).

Acknowledgements. We would like to thank Nageswari Shanmugalingam, David Drasin and David Swanson for valuable comments on the manuscript.

This paper was completed while the first author was visiting Purdue University in 2007–2008. He acknowledges the people at the Department of Mathematics for hospitality and stimulating conversations.

2. PRELIMINARIES

Throughout the paper $X = (X, d, \mu)$ is an unbounded complete metric space endowed with a metric d and a positive complete Borel regular

measure μ such that $0 < \mu(B(x, r)) < \infty$ for all balls $B(x, r) := \{x_0 \in X : d(x, x_0) < r\}$ in X ; and if $B = B(x, r)$, then $CB = B(x, Cr)$ for $C > 0$. We denote by $B_m(x, r)$ the open ball in \mathbb{R}^m about x with radius r .

We also assume that

- H1. X supports a weak $(1, p_0)$ -Poincaré inequality, where either $p_0 > m$ or $p_0 \geq m = 1$;
- H2. the measure μ is doubling and satisfies the lower mass bound (3.4).

In what follows, and if not otherwise stated, $1 \leq m \leq Q$. We clarify these assumptions and concepts below.

The measure μ is said to be *doubling*, if there exists a constant $C_\mu \geq 1$, called the *doubling constant* of μ , such that

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)),$$

for all $x \in X$ and $r > 0$. We note that the doubling condition implies that for every $x \in X$ and $r > 0$ one has for $\lambda \geq 1$

$$\mu(B(x, \lambda r)) \leq C \lambda^Q \mu(B(x, r)),$$

where $Q = \log_2 C_\mu$, and the constant C depends only on C_μ . The exponent Q serves as a dimension of the doubling measure μ ; we emphasize that it need not to be an integer.

Given a subset E of X , for each $\varepsilon > 0$ we define the premeasure

$$\mathcal{H}_\varepsilon^Q(E) = \inf_{B \in \mathcal{B}} \sum (\text{diam } B)^Q,$$

where the infimum is taken over all covers \mathcal{B} of E by (closed) balls of diameter at most ε (see, e.g., Heinonen [12]). We may define the *Hausdorff Q -measure* of E as

$$\mathcal{H}^Q(E) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon^Q(E).$$

The *upper Q -density* of a finite Borel regular measure μ at x is defined by

$$\Theta_Q^*(\mu, x) = \limsup_{r \rightarrow 0+} \frac{\mu(B(x, r))}{\omega(Q)r^Q},$$

where $\omega(Q)$ is a constant depending only on Q . We record that if for all $x \in E$, E a Borel set in X , the upper Q -density of μ at x is at least α , $\alpha \in (0, \infty)$, then

$$\mu(A) \geq \alpha \mathcal{H}^Q(E \cap A) \quad \text{for } A \subset X.$$

Also, if the upper Q -density of μ at $x \in E$ is at most α we obtain

$$\mu(E \cap A) \leq 2^Q \alpha \mathcal{H}^Q(E \cap A) \quad \text{for } A \subset X.$$

See Federer [9, 2.10.19].

Note that a Vitali-type covering theorem is available in metric spaces with a doubling measure. More precisely, from a given a family of balls with an upper bound for radii covering a set $E \subset X$ we can select a pairwise disjoint subfamily $\{B_i\} := \{B(x_i, r_i)\}$ such that

$$E \subset \bigcup_i B(x_i, 5r_i),$$

see e.g. Heinonen [12], Mattila [26]. Since a complete metric space with a doubling measure is separable the subfamily is countable.

In this paper, a *path* in X is a continuous mapping from a compact interval to X . A path can thus be parametrized by arc length ds .

A nonnegative Borel function g on X is an *upper gradient* of an extended real valued function f on X if for all rectifiable paths γ joining points x and y in X we have

$$(2.1) \quad |f(x) - f(y)| \leq \int_{\gamma} g \, ds.$$

whenever both $f(x)$ and $f(y)$ are finite, and $\int_{\gamma} g \, ds = \infty$ otherwise. See Cheeger [4] and Shanmugalingam [29] for a discussion on upper gradients.

If g is a nonnegative measurable function on X and if (2.1) holds for p -almost every path, then g is a *weak upper gradient* of f . By saying that (2.1) holds for p -almost every path we mean that it fails only for a path family with zero p -modulus (see, e.g., [29]).

If u has an upper gradient in $L^p(X)$, then it has a *minimal weak upper gradient* $g_f \in L^p(X)$ in the sense that for every weak upper gradient $g \in L^p(X)$ of f , $g_f \leq g$ μ -almost everywhere (a.e.), see Corollary 3.7 in Shanmugalingam [30]. The minimal weak upper gradient can be obtained by the formula

$$g_f(z) := \inf_g \limsup_{r \rightarrow 0^+} \frac{1}{\mu(B(z, r))} \int_{B(z, r)} g \, d\mu,$$

where the infimum is taken over all upper gradients $g \in L^p(X)$ of f , see Lemma 2.3 in J. Björn [3].

We define Sobolev spaces on metric spaces following Shanmugalingam [29]. Let $\Omega \subseteq X$ be nonempty and open. Whenever $u \in L^p(\Omega)$, let

$$\|u\|_{N^{1,p}(\Omega)} := \|u\|_{1,p} := \left(\int_{\Omega} |u|^p \, d\mu + \int_{\Omega} g_u^p \, d\mu \right)^{1/p}.$$

The *Newtonian space* on Ω is the quotient space

$$N^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$. The space $N^{1,p}(\Omega)$ is a Banach space and a lattice. If $\Omega \subset \mathbb{R}^n$ is open, then $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ as Banach spaces. For these and other properties of Newtonian spaces we refer to [29].

The class $N^{1,p}(\Omega; \mathbb{R}^m)$, $1 \leq m \leq Q$, consists of those mappings $u : \Omega \rightarrow \mathbb{R}^m$ whose component functions each belong to $N^{1,p}(\Omega) = N^{1,p}(\Omega; \mathbb{R})$. Qualitative properties like Lebesgue points, density of Lipschitz functions, quasicontinuity, etc. may be investigated componentwise.

The *variational p -capacity* of a Borel set $E \subset X$ is the number

$$\text{cap}_p(E) = \inf \int_X g_u^p d\mu,$$

where the infimum is taken over all weak upper gradients g_u of some $u \in N^{1,p}(X)$ such that $u \geq 1$ on E . If such functions do not exist, we set $\text{cap}_p(E) = \infty$. Observe that if $\mu(X) < \infty$ the constant function will do as a test function, thus, all sets are of zero capacity.

Under our assumptions, the variational p -capacity is a Choquet capacity.

There are several definitions for capacities. Let $\Omega \subset X$ is an open set. The *relative p -capacity* of a set $E \subset \Omega$ is the number

$$\text{Cap}_p(E, \Omega) = \inf \int_\Omega g_u^p d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E .

See, e.g., Kinnunen–Martio [18], [19] for a discussion on capacities on metric spaces.

We say that X supports a *weak $(1, p)$ -Poincaré inequality* if there exist constants $C > 0$ and $\tau \geq 1$ such that for all balls $B(z, r) \subset X$, all measurable functions f on X and for all weak upper gradients g_f of f ,

$$\int_{B(z,r)} |f - f_{B(z,r)}| d\mu \leq Cr \left(\int_{B(z,\tau r)} g_f^p d\mu \right)^{1/p},$$

where $f_{B(z,r)} := \int_{B(z,r)} f d\mu / \mu(B(z,r))$.

It is known, see e.g. Heinonen [13, Proposition 10.9] that the embedding $N^{1,p}(X) \rightarrow L^p(X)$ is not surjective if and only if X there exists a path family in X with a positive p -modulus. Moreover, the validity of a Poincaré inequality can sometimes be stated in terms of p -modulus. More precisely, to require that (2) holds in X is to require that the p -modulus of paths between every pair of distinct points of the space is sufficiently large, see Theorem 2 in Keith [16].

Under our assumptions it follows that Lipschitz functions are dense in $N^{1,p_0}(X)$. Again, in $N^{1,p_0}(X; \mathbb{R}^m)$ this should be understood componentwise. See Shanmugalingam [29, Theorem 4.1] and also Hajlasz [10, Theorem 5] for the following Luzin-type theorem.

Theorem 2.1. *Let $u \in N^{1,p_0}(X; \mathbb{R}^m)$. Then for every $\varepsilon > 0$ there is a Lipschitz function φ on X such that*

$$\mu(\{x \in X : u(x) \neq \varphi(x)\}) < \varepsilon,$$

and $\|u - \varphi\|_{1,p_0} < \varepsilon$.

In addition to this, functions in $N^{1,p_0}(X)$ are quasicontinuous, see Björn et al. [2]. A function is said to be quasicontinuous if for there exists an open set G with arbitrarily small capacity such that the function is continuous on the complement of G . A mapping $N^{1,p_0}(X; \mathbb{R}^m)$ is said to be quasicontinuous if each of its component functions is quasicontinuous. In the Euclidean setting, $N^{1,p_0}(\mathbb{R}^m)$ is the refined Sobolev space as defined on p. 96 of Heinonen et al. [14].

3. CONDITION (N)

Given a real number α , we denote by $\mathcal{H}_\infty^\alpha(E)$ the Hausdorff α -content of $E \subset X$ defined as

$$\mathcal{H}_\infty^\alpha(E) = \inf \sum_{B \in \mathcal{B}} (\text{diam } B)^\alpha,$$

where the infimum is taken over all covers \mathcal{B} of E by (closed) balls. Clearly $\mathcal{H}_\infty^\alpha(E) \leq \mathcal{H}^\alpha(E)$.

A mapping $f : X \rightarrow \mathbb{R}^d$, $d \geq Q$, or $f : X \rightarrow X \times \mathbb{R}^m$, is said to satisfy *(Luzin's) condition (N)* if $\mathcal{H}^Q(f(E)) = 0$ whenever $E \subset X$ satisfies $\mu(E) = 0$.

We denote by $\bar{f} : X \rightarrow X \times \mathbb{R}^m$ the *graph mapping* of f :

$$\bar{f}(x) = (x, f(x)), \quad x \in X,$$

and $\mathcal{G}_f(X)$ is the *graph* of f over X defined by

$$\mathcal{G}_f(X) = \{(x, f(x)) : x \in X\} \subset X \times \mathbb{R}^m.$$

We, furthermore, denote by $\pi : X \times \mathbb{R}^m \rightarrow X$ the projection

$$\pi(x, x_1, x_2, \dots, x_m) = x$$

whenever $x \in X$ and $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$, and by $\rho : X \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ the projection

$$\rho(x, x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_m)$$

whenever $x \in X$ and $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$. Note that $\text{Lip}(\pi) = \text{Lip}(\rho) = 1$.

Remark 3.1. Let $f : X \rightarrow \mathbb{R}^m$ be measurable and f^* a Borel measurable representative of f . (The regularity of the measure μ implies that if f is measurable, then there exist Borel measurable functions f_* , f^* such that $f_* \leq f \leq f^*$ and $f_* = f^*$ μ -a.e.) Thus the graph $\mathcal{G}_{f^*}(X)$ of f^* is a Borel subset of $X \times \mathbb{R}^m$. Then [20, Theorem 2, p. 385] implies that the projection $\pi(\mathcal{G}_{f^*}(X) \cap E)$ is Borel measurable for every Borel set $E \subset X \times \mathbb{R}^m$. Since f and f^* agree up to a set of μ measure zero, so do sets $\pi(\mathcal{G}_{f^*}(X) \cap E)$ and $\pi(\mathcal{G}_f(X) \cap E)$, implying that $\pi(\mathcal{G}_f(X) \cap E)$ is measurable.

The proof of the following lemma is an easy adaptation of the proof by Malý et al. in [24].

Lemma 3.2. *Suppose $m \leq d \leq m + Q$ and let $E \subset X \times \mathbb{R}^m$. Then*

$$\mathcal{H}_\infty^d(E) \leq C(\text{diam } E)^m \mathcal{H}_\infty^{d-m}(\pi(E)),$$

where the constant C depends only on m .

Proof. We may assume that $\text{diam } E < \infty$. By a Vitali-type covering theorem we can select a pairwise disjoint family of balls $\{B_i\} := \{B(x_i, r_i)\}$ such that

$$\pi(E) \subset \bigcup_i B(x_i, 5r_i).$$

For each i let M_i denote the greatest integer satisfying

$$(3.1) \quad (M_i - 1) \text{diam } B_i < \text{diam } E.$$

Since $E \cap \pi^{-1}(B_i)$ is bounded in $X \times \mathbb{R}^m$, it can be contained in a large enough cylinder of the form $B_i \times Q_i \subset X \times \mathbb{R}^m$, where Q_i is a cube in \mathbb{R}^m with side length $\text{diam } E$. Since $M_i \text{diam } B_i \geq \text{diam } E$ (recall that M_i is the greatest integer satisfying (3.1)), Q_i may be covered by M_i^m cubes $\{Q_{i,j}\}$ of sidelength $\text{diam } B_i$. We get

$$\begin{aligned} \mathcal{H}_\infty^d(E \cap \pi^{-1}(B_i)) &\leq \sum_{j=1}^{M_i^m} (\text{diam } B_i \times Q_{i,j})^d \leq M_i^m (\text{diam } B_i)^d \\ &\leq (M_i \text{diam } B_i)^m (\text{diam } B_i)^{d-m} \\ &\leq (\text{diam } E + \text{diam } B_i)^m (\text{diam } B_i)^{d-m}. \end{aligned}$$

Since $\text{diam } B_i \leq \text{diam } \pi(E) \leq \text{diam } E$, summing over i shows that

$$\mathcal{H}_\infty^d(E) \leq C(\text{diam } E)^m \sum_{i=1}^{\infty} (\text{diam } B_i)^{d-m}.$$

The result follows by taking the infimum over all coverings $\{B_i\}$. \square

We now state a general criterion for condition (N) similar to that of Radó and Reichelderfer, see [27, V.3.6] and Malý [21]. This result was proved in the Euclidean case by Malý et al. in [24].

Theorem 3.3. *Let f be a Borel measurable function and suppose that there is $\theta \in L^1_{\text{loc}}(X)$ such that*

$$(3.2) \quad \mathcal{H}_\infty^{Q-m}(\pi(\mathcal{G}_f(X) \cap B(z, r))) \leq \frac{1}{r^m} \int_{\pi(\mathcal{G}_f(X) \cap B(z, 4r))} \theta(x) d\mu(x)$$

for all $z \in X \times \mathbb{R}^m$ and $r > 0$. Then there exist a constant $C = C(Q, m)$ such that

$$\mathcal{H}^Q(\bar{f}(E)) \leq C \int_E \theta(x) d\mu(x)$$

for all measurable set $E \subset X$. In particular, \bar{f} satisfies condition (N).

Proof. Define a set function σ on $X \times \mathbb{R}^m$ by

$$\sigma(E) = \int_{\pi(\mathcal{G}_f(X) \cap E)} \theta \, d\mu.$$

Lemma 3.2 with $d = Q$ and (3.2) give us for any $z \in X \times \mathbb{R}^m$ and $r > 0$

$$\begin{aligned} (3.3) \quad \mathcal{H}_\infty^Q(\mathcal{G}_f(X) \cap B(z, r)) &\leq \mathcal{H}_\infty^{Q-m}(\pi(\mathcal{G}_f(X) \cap B(z, r))) \\ &\leq C \int_{\pi(\mathcal{G}_f(X) \cap B(z, 4r))} \theta \, d\mu \\ &\leq C\sigma(B(z, 4r)). \end{aligned}$$

Since for \mathcal{H}^Q -almost every $z \in \mathcal{G}_f(X)$, see Federer [7, Lemma 10.1],

$$\limsup_{r \rightarrow 0+} \frac{\mathcal{H}_\infty^Q(\mathcal{G}_f(X) \cap B(z, r))}{\omega(Q)r^Q} \geq C,$$

it follows from (3.3) that

$$\limsup_{r \rightarrow 0+} \frac{\sigma(B(z, r))}{\omega(Q)r^Q} \geq C$$

for \mathcal{H}^Q -almost every $z \in \mathcal{G}_f(X)$. Remark 3.1 implies that σ is a measure on the Borel Σ -algebra of $X \times \mathbb{R}^m$, and it may be extended to a regular Borel outer measure σ^* on all of $X \times \mathbb{R}^m$ in the usual way

$$\sigma^*(A) := \inf\{\sigma(E) : A \subset E, E \subset X \text{ is a Borel set}\}.$$

Since $\theta \in L^1_{\text{loc}}(X)$ it follows that σ^* is a Radon measure on $X \times \mathbb{R}^m$. Therefore,

$$\mathcal{H}^Q(E) \leq C\sigma^*(E)$$

for all $E \subset \mathcal{G}_f(X)$. Finally, given a measurable set $E \subset X$, choose a Borel set G with $E \subset G$. Then $\bar{f}(E) \subset G \times \mathbb{R}^m$, $G \times \mathbb{R}^m$ is a Borel set, and

$$\mathcal{H}^Q(\bar{f}(E)) \leq C\sigma^*(\bar{f}(E)) \leq C\sigma(G \times \mathbb{R}^m) = C \int_G \theta \, d\mu.$$

The proof is completed by taking the infimum over all such G 's.

Now, if $E \subset X$ such that $\mu(E) = 0$ then it readily follows that $\mathcal{H}^Q(\bar{f}(E)) = 0$. This completes the proof. \square

From here on out our objective is to provide an application for the general criterion for condition **(N)**. Indeed, we shall show that the graph mapping of a Newtonian function satisfies condition **(N)**.

We start with a few auxiliary lemmas. The following capacity estimate relates the 1-capacity with the Hausdorff content. In the metric measure space setting it was proved by Kinnunen et al. in [17, Theorem 3.6].

Lemma 3.4. *Suppose that there exists a constant $C > 0$, depending only on C_μ , such that the measure μ satisfies the lower mass bound*

$$(3.4) \quad \mu(B(x, r)) \geq Cr^Q$$

for all $x \in X$ and $0 < r < \text{diam } X$. Let $E \subset X$ be a Borel set. Then

$$\mathcal{H}_\infty^{Q-1}(E) \leq C \text{cap}_1(E),$$

where the constant depends only on the doubling constant C_μ and the constants in the weak $(1, 1)$ -Poincaré inequality.

Remark 3.5. It follows that if $u \in N^{1,1}(X)$ such that $u \geq 1$ on E and g_u is a weak upper gradient of u , Lemma 3.4 implies that

$$\mathcal{H}_\infty^{Q-1}(E) \leq C \int_X g_u d\mu,$$

where the constant C is from Lemma 3.4.

We also have the following relation between the p_0 -capacity and the Hausdorff content when $p_0 > 1$.

Lemma 3.6. *Suppose that there exists a constant $C > 0$, depending only on C_μ , such that the measure μ satisfies the lower mass bound (3.4) for all $x \in X$ and $0 < r < \text{diam } X$. Let $E \subset X$ be a Borel set and suppose that $t > Q - p_0$. Then*

$$\mathcal{H}_\infty^t(E \cap B(x_0, r)) \leq Cr^{t-Q+p_0} \text{Cap}_{p_0}(E \cap B(x_0, r), B(x_0, 2r)),$$

where $x_0 \in X$, $r > 0$, and $C = C(p_0, Q, t, C_\mu)$.

For the proof of this result we refer to Costea [5].

Remark 3.7. If $u \in N_0^{1,p_0}(B(x_0, 2r))$ such that $u \geq 1$ on $E \cap B(x_0, r)$, g_u is a weak upper gradient of u , and $t = Q - m$, where $1 \leq m < p_0$, we obtain

$$\mathcal{H}_\infty^{Q-m}(E \cap B(x_0, r)) \leq Cr^{p_0-m} \int_{B(x_0, 2r)} g_u^{p_0} d\mu,$$

where the constant C is as in Lemma 3.6.

Next we provide an application of the general criterion for condition (N) stated in Theorem 3.3.

Theorem 3.8. *Let $u \in N^{1,p_0}(X; \mathbb{R}^m)$, where either $p_0 > m$ or $p_0 \geq m = 1$. Then the graph mapping \bar{u} satisfies condition (N).*

Proof. It is sufficient to verify the assumptions of Theorem 3.3 with some locally integrable function θ .

Let $p_0 > m$ and, to this end, fix a point $z \in X \times \mathbb{R}^m$ and $r > 0$. Writing $z = (x_0, y_0) \in X \times \mathbb{R}^m$ we obtain

$$\mathcal{G}_u(X) \cap B(z, r) \subset (\mathcal{G}_u(X) \cap (B(x_0, r) \times B_m(y_0, r))).$$

Thus we also have that

$$\pi(\mathcal{G}_u(X) \cap B(z, r)) \subset (B(x_0, r) \cap u^{-1}(B_m(y_0, r))),$$

Hence $|u(x)| \leq r$ for each $x \in B(x_0, r) \cap u^{-1}(B_m(y_0, r))$. Let us define

$$v(x) = 2 \max \left\{ 1 - \frac{|u(x) - u(x_0)|}{2r}, 0 \right\},$$

and let $\eta \in \text{Lip}(X)$ such that $\eta = 1$ on $B(x_0, r)$, $\eta = 0$ in $X \setminus B(x_0, 2r)$, $0 \leq \eta \leq 1$, and $g_\eta \leq C/r$. Then $v\eta \geq 1$ on $B(x_0, r) \cap u^{-1}(B_m(y_0, r))$, and $v\eta \in N_0^{1,p_0}(B(x_0, 2r))$, thus, Lemma 3.6 implies that

$$\begin{aligned} \mathcal{H}_\infty^{Q-m}(\pi(\mathcal{G}_u(X) \cap B(z, r))) &\leq \mathcal{H}_\infty^{Q-m}(B(x_0, r) \cap u^{-1}(B_m(y_0, r))) \\ &\leq Cr^{p_0-m} \int_{B(x_0, 2r) \cap \{v>0\}} g_{v\eta}^{p_0} d\mu \\ &\leq Cr^{p_0-m} \int_{B(x_0, 2r) \cap u^{-1}(B_m(y_0, 2r))} (v^{p_0} g_\eta^{p_0} + \eta^{p_0} g_v^{p_0}) d\mu \\ &\leq Cr^{-m} \int_{B(x_0, 2r) \cap u^{-1}(B_m(y_0, 2r))} (1 + g_u^{p_0}) d\mu. \end{aligned}$$

Since

$$B(x_0, 2r) \cap u^{-1}(B_m(y_0, 2r)) \subset \pi(\mathcal{G}_u(X) \cap B(z, 4r)),$$

above reasoning gives us that

$$\mathcal{H}_\infty^{Q-m}(\pi(\mathcal{G}_u(X) \cap B(z, r))) \leq \frac{C}{r^m} \int_{\pi(\mathcal{G}_u(X) \cap B(z, 4r))} (1 + g_u^{p_0}) d\mu.$$

This verifies the assumptions of Theorem 3.3 with $\theta = 1 + g_u^{p_0}$, and thus concludes the proof when $p_0 > m$. The case $p_0 \geq m = 1$ is dealt with similar argument, albeit using Lemma 3.6 we apply Lemma 3.4 instead. This completes the proof. \square

4. RECTIFIABLE SETS AND COAREA PROPERTY

Next we study rectifiability properties of the graph of a Newtonian function. We start by recalling that a set $E \subset X \times \mathbb{R}^m$ is said to be *countably \mathcal{H}^Q -rectifiable* if there exists subsets $E_i \subset X$ and at most countable collection of Lipschitz mappings $f_i : E_i \rightarrow \mathbb{R}^m$ with the property that

$$\mathcal{H}^Q \left(E \setminus \bigcup_{i=1}^{\infty} \overline{f_i(E_i)} \right) = 0.$$

We have the following theorem which is a fairly straightforward consequence of Theorems 2.1 and 3.8.

Theorem 4.1. *Let $u \in N^{1,p_0}(X; \mathbb{R}^m)$, where $p_0 > m$ or $p_0 \geq m = 1$. Then the graph $\mathcal{G}_u(X)$ is countably \mathcal{H}^Q -rectifiable.*

Proof. Let $u \in N^{1,p_0}(X; \mathbb{R}^m)$, where $p_0 > m$ (the other case is treated similarly), and $\varepsilon = 2^{-i}$, $i = 1, 2, \dots$. Due to Theorem 2.1 for each $i = 1, 2, \dots$ there is a Lipschitz function $l_i : X \rightarrow \mathbb{R}^m$ such that $\mu(X \setminus E_i) < 2^{-i}$, where $E_i = \{x \in X \setminus E_{i-1} : u(x) = l_i(x)\}$, and $E_1 =$

$\{x \in X : u(x) = l_1(x)\}$ such that $\mu(X \setminus E_1) < 2^{-1}$. This gives us that $\mu(X \setminus \bigcup_i E_i) = 0$. Moreover, since \bar{u} satisfies condition (N) by Theorem 3.8 we have

$$\mathcal{H}^Q\left(\mathcal{G}_u(X) \setminus \bigcup_{i=1}^{\infty} \bar{l}_i(E_i)\right) = 0,$$

and the claim follows. \square

We say that a function $f : X \rightarrow \mathbb{R}^m$ satisfies the *t-coarea property*, or the *condition co-N*, for some $t > 0$ in X if for each set $E \subset X$ such that $\mu(E) = 0$ and for \mathcal{H}^m -almost every $y \in \mathbb{R}^m$ we have $\mathcal{H}^t(E \cap f^{-1}(y)) = 0$.

The next lemma, due to Federer [9, 2.10.25, 2.10.26], is a version of Eilenberg's inequality [6]. Consult also Malý [22, Theorem 13.2].

Lemma 4.2 (Eilenberg inequality). *Let $f : X \rightarrow \mathbb{R}^m$ be Lipschitz, $A \subset X$, $0 \leq k < \infty$, and $0 \leq h < \infty$, then*

$$\int_{\mathbb{R}^m}^* \mathcal{H}^k(A \cap f^{-1}(y)) d\mathcal{H}^h(y) \leq C \text{Lip}(f)^h \mathcal{H}^{k+h}(A),$$

where C is a constant depending only on k and h .

The symbol \int^* denotes the upper integral, see, e.g., Federer [9].

It now follows from Eilenberg's inequality that a Newtonian function satisfies the coarea property in X .

Theorem 4.3. *Let $u \in N^{1,p_0}(X; \mathbb{R}^m)$, where $p_0 > m$ or $p_0 \geq m = 1$. Then u satisfies the $Q - m$ -coarea property.*

Proof. Let $E \subset X$ so that $\mu(E) = 0$. Apply the Eilenberg inequality with $A = \bar{u}(E)$, $f = \rho$, $h = m$, and $k = Q - m$. Then, thanks to Theorem 3.8

$$\int_{\mathbb{R}^m}^* \mathcal{H}^{Q-m}(\bar{u}(E) \cap \rho^{-1}(y)) d\mathcal{H}^m(y) \leq C \mathcal{H}^Q(\bar{u}(E)) = 0.$$

Since $\bar{u}(E) \cap \rho^{-1}(y) = \{(x, u(x)) \in X \times \mathbb{R}^m : x \in E, u(x) = y\}$, it follows that $\pi(\bar{u}(E) \cap \rho^{-1}(y)) = E \cap u^{-1}(y)$. The fact that Hausdorff measures do not increase on projection, gives us that

$$\int_{\mathbb{R}^m}^* \mathcal{H}^{Q-m}(E \cap u^{-1}(y)) d\mathcal{H}^m(y) = 0,$$

so that $\mathcal{H}^{Q-m}(E \cap u^{-1}(y)) = 0$, thus completing the proof. \square

We close the section with the following result.

Proposition 4.4. *Let $u \in N^{1,p_0}(X; \mathbb{R}^m)$, where $p_0 > m$ or $p_0 \geq m = 1$, and $S \subset X$ be a countably \mathcal{H}^Q -rectifiable set. Then for \mathcal{H}^m -a.e. y the set $u^{-1}(y) \cap S$ is countably \mathcal{H}^{Q-m} -rectifiable.*

Proof. Let $S \subset X$ be a countably \mathcal{H}^Q -rectifiable set and $(l_i)_{i \geq 1}$ a set of Lipschitz maps from X to \mathbb{R}^m . Then, for each $i = 1, 2, \dots$, Ambrosio–Kirchheim [1, Theorem 9.4] implies for \mathcal{H}^m -a.e. $y \in \mathbb{R}^m$ that the set $u^{-1}(y) \cap S \cap E_i = l_i^{-1}(y) \cap S \cap E_i$ is countably \mathcal{H}^{Q-m} -rectifiable, where $\mu(X \setminus E_i) < 2^{-i}$ and sets E_i , $i = 1, 2, \dots$, are as in the proof of Theorem 4.1. The proof follows by observing that thanks to the coarea property of u , $\mathcal{H}^{Q-m}(u^{-1}(y) \cap S \cap E) = 0$ for \mathcal{H}^m -a.e. $y \in \mathbb{R}^m$ whenever $\mu(E) = 0$, and recalling that $\mu(X \setminus \bigcup_i E_i) = 0$. \square

5. THE COAREA FORMULA

In this section we prove the coarea formula for Newtonian functions. We start by defining for a measurable $f : X \rightarrow \mathbb{R}^m$ and for given $x \in X$ the *coarea factor of f at x* by

$$\begin{aligned} C_m f(x) & \left(= \limsup_{r \rightarrow 0^+} \frac{\nu(B(x, r))}{\mathcal{H}^Q(B(x, r))} \right) \\ & := \limsup_{r \rightarrow 0^+} \frac{1}{\mathcal{H}^Q(B(x, r))} \int_{\mathbb{R}^m} \mathcal{H}^{Q-m}(B(x, r) \cap f^{-1}(y)) d\mathcal{H}^m(y). \end{aligned}$$

We want to record a few simple observations. If $f \in \text{Lip}(X)$, then by the Eilenberg inequality, Lemma 4.2, we obtain that for every $x \in X$

$$C_m f(x) \leq C \text{Lip}(f)^m,$$

where C depends only on Q and m . Moreover, the measure ν is absolutely continuous with respect to \mathcal{H}^Q . Observe also that when $X = \mathbb{R}^n$ then $C_m(f)(x) = J_m f(x)$, the m -dimensional Jacobian of f , by Federer [8] (see the proof of Theorem 3.1).

We can now state the coarea formula in our setting.

Theorem 5.1. *Let $\Omega \subset X$ be an open set and $u \in N^{1,p_0}(\Omega; \mathbb{R}^m)$, where $p_0 > m$ or $p_0 \geq m = 1$. Then for every Borel function $\theta : S \rightarrow [0, \infty]$ one has*

$$\int_S \theta(x) C_m u(x) d\mathcal{H}^Q(x) = \int_{\mathbb{R}^m} \left(\int_{u^{-1}(y)} \theta(x) d\mathcal{H}^{Q-m}(x) \right) d\mathcal{H}^m(y),$$

where $S \subset X$ is a countably \mathcal{H}^Q -rectifiable set.

Proof. Let $S \subset X$ be a countably \mathcal{H}^Q -rectifiable set. From Theorem 9.4 in Ambrosio and Kirchheim [1], we have for every Borel function $\theta : S \rightarrow [0, \infty]$ and $i = 1, 2, \dots$,

$$\int_{S \cap E_i} \theta(x) C_m u(x) d\mathcal{H}^Q(x) = \int_{\mathbb{R}^m} \left(\int_{u^{-1}(y)} \theta(x) d\mathcal{H}^{Q-m}(x) \right) d\mathcal{H}^m(y),$$

where $E_1 = \{x \in X : u(x) = l_1(x)\}$, and $E_i = \{x \in X \setminus E_{i-1} : u(x) = l_i(x)\}$, $i = 2, 3, \dots$, where l_i is a Lipschitz function on X . Recall that $\mu(X \setminus \bigcup_{i=1}^\infty E_i) = 0$. Due to Theorem 4.3 u satisfies the $Q - m$ -coarea

property, hence, for \mathcal{H}^m -a.e. $y \in \mathbb{R}^m$ we have $\mathcal{H}^{Q-m}(E \cap u^{-1}(y)) = 0$ whenever $\mu(E) = 0$. The coarea formula follows. \square

Let $f : X \rightarrow \mathbb{R}^m$ be linear. Ambrosio and Kirchheim, defined the coarea factor to be the unique constant, such that

$$C_m(f)\mathcal{H}^Q(E) = \int_{\mathbb{R}^m} \mathcal{H}^{Q-m}(E \cap f^{-1}(y)) d\mathcal{H}^m(y)$$

for all Borel measurable set $E \subset X$. (See Definition 9.1 in [1]. They proved that for every Borel function $\theta : S \rightarrow [0, \infty]$

$$(5.1) \quad \int_S \theta(x) C_m(d^S g_x) d\mathcal{H}^Q(x) = \int_{\mathbb{R}^m} \left(\int_{u^{-1}(y)} \theta(x) d\mathcal{H}^{Q-m}(x) \right) d\mathcal{H}^m(y),$$

where $d^S g_x$ is the approximate tangential differential, see [1]. Since in the metric space setting we are not able to use the linear approximation we need the local definition for the coarea factor. However, if we let above $\theta = \chi_{B(x,r)}$ and $g \in \text{Lip}(X; \mathbb{R}^m)$. Then differentiating both sides of (5.1) with respect to $B(x, r)$ yields to

$$\begin{aligned} C_m(d^S g_x)(x) &= \limsup_{r \rightarrow 0+} \int_{B(x,r)} C_m(d^S g_x) d\mathcal{H}^Q(x) \\ &= \limsup_{r \rightarrow 0+} \frac{1}{\mathcal{H}^Q(B(x, r))} \int_{\mathbb{R}^m} \mathcal{H}^{Q-m}(B(x, r) \cap g^{-1}(y)) d\mathcal{H}^m(y) \\ &= C_m(g)(x) \end{aligned}$$

for a.e. every $x \in X$. Thus the coarea factors are equal for a.e. x .

6. CONDITION (N) CONTINUED

We close this paper by proving condition (N) for Newtonian functions $N^{1,p_0}(\Omega; \mathbb{R}^m)$, where $p_0 > Q$. In this section $m > Q$. The approach is closely related to Malý's in [21].

Let $\Omega \subset X$ be open and bounded. Following Malý we say that a mapping $f : \Omega \rightarrow \mathbb{R}^m$ is Q -absolutely continuous if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every disjoint finite family $\{B_i\}$ of closed balls in Ω we have

$$\sum_i \mu(B_i) < \delta \quad \Rightarrow \quad \sum_i (\text{osc}_{B_i} f)^Q < \varepsilon,$$

where $\text{osc}_{B_i} f$ is the oscillation of f over the ball B_i , which is the diameter of the image $f(B_i)$. In particular, Q -absolute continuity of a mapping $f = (f_1, \dots, f_m)$ is equivalent to Q -absolute continuity of each component function f_i , $i = 1, \dots, m$.

Also following Malý we say that a function $f : \Omega \rightarrow \mathbb{R}^m$ satisfies the Rado–Reichelderfer condition, or *condition* (RR) for short, if there

exists a nonnegative function $\varphi \in L^1_{\text{loc}}(\Omega)$ such that

$$(\text{osc}_B f)^Q \leq \int_{5\tau B} \varphi d\mu$$

for every $B \subset \Omega$ with $5\tau B \subset \Omega$, where $\tau \geq 1$ is the dilation constant in the weak $(1, p_0)$ -Poincaré inequality. A condition similar to this was used in [27, V.3.6] as a sufficient condition for the condition **(N)** and for a.e. differentiability.

The following proposition readily follows from results in [21], see theorems 3.1 and 4.1.

Proposition 6.1. *Let $u \in N^{1,p_0}(\Omega; \mathbb{R}^m)$, where $p_0 > Q$, $m \geq 1$. Then u is Q -absolutely continuous.*

Proof. By the imbedding theorem in Hajłasz–Koskela [11, Theorem 5.1], one has

$$\text{osc}_B u \leq C r^{1-Q/p_0} \left(\int_{5\tau B} g_u^{p_0} \right)^{1/p_0}.$$

It follows from Young's inequality and (3.4) that u satisfies condition (RR) with $\varphi = C(1 + g_u^{p_0})$, where $0 < C < \infty$ is a uniform constant depending only on p_0 , Q , τ , C_μ , and the constant in the weak $(1, p_0)$ -Poincaré inequality. The proof follows by observing that condition (RR) implies the Q -absolute continuity of u . \square

Corollary 6.2. *Let $u \in N^{1,p_0}(\Omega; \mathbb{R}^m)$, where $p_0 > Q$, and $m > Q \geq 1$. Then u satisfies condition **(N)**.*

Proof. Let $E \subset \Omega$ be a μ -null set and $\varepsilon > 0$. Then there exists an open set $G \subset \Omega$ such that $E \subset G$ and $\mu(G) < \delta$. Since u is Q -absolutely continuous by Proposition 6.1, for each $x \in E$ there is $r_x > 0$ so that $B(x, r_x) \subset G$ and $\text{osc}_{B(x, r_x)} u \leq \varepsilon/4$. By a Vitali-type covering theorem we can select a pairwise disjoint subfamily $\{B(u(x_i), \text{osc}_{B(x_i, r_{x_i})} u)\}$ such that

$$u(E) \subset \bigcup_i B_m(u(x_i), 5 \text{osc}_{B(x_i, r_{x_i})} u).$$

Thus one has

$$\mathcal{H}_\varepsilon^Q(u(E)) \leq \sum_i (\text{diam } B_i)^Q \leq C \sum_i (\text{osc}_{B(x_i, r_{x_i})} u)^Q \leq C\varepsilon,$$

and the claim follows by passing to the limit $\varepsilon \rightarrow 0+$. \square

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